QUANTUM LIMITS IN SPACE-TIME MEASUREMENTS

Marc-Thierry Jaekel† and Serge Reynaud‡

†Laboratoire de Physique Théorique de l'Ecole Normale Supérieure[1] (CNRS),
24 rue Lhomond, F75231 Paris Cedex 05, France
‡Laboratoire Kastler-Brossel[2] (UPMC-ENS-CNRS), case 74,
4 place Jussieu, F75252 Paris Cedex 05, France

Abstract

Quantum fluctuations impose fundamental limits on measurement and space-time probing. Although using optimised probe fields can allow to push sensitivity in a position measurement beyond the "standard quantum limit", quantum fluctuations of the probe field still result in limitations which are determined by irreducible dissipation mechanisms. Fluctuation-dissipation relations in vacuum characterise the mechanical effects of radiation pressure vacuum fluctuations, which lead to an ultimate quantum noise for positions. For macroscopic reflectors, the quantum noise on positions is dominated by gravitational vacuum fluctuations, and takes a universal form deduced from quantum fluctuations of space-time curvatures in vacuum. These can be considered as ultimate space-time fluctuations, fixing ultimate quantum limits in space-time measurements.

PACS numbers: 12.20 Ds 03.70 42.50 Lc

LPTENS 95/13
May 1995

---

[1] Unité propre du Centre National de la Recherche Scientifique, associée à l'Ecole Normale Supérieure et à l'Université de Paris Sud.

[2] Unité de l'Ecole Normale Supérieure et de l'Université Pierre et Marie Curie, associée au Centre National de la Recherche Scientifique.

I. Introduction

Interest in high sensitivity measurements of position has been recently revived, under the impulse of projects for detecting gravitational waves [1]. The required very high level of sensitivity has led to consider the role of limitations due to quantum fluctuations [2]. Limits imposed by quantum mechanics give rise to practical and theoretical problems. When disregarding noise sources (like seismic or thermal noises), which although dominant can be minimised in principle, fundamental limitations subsist which cannot be bypassed.

An early argument related to Heisenberg's microscope asserts that measurement of a quantity q with a precision (mean square deviation) Δq must induce a perturbation of its conjugate variable p of an amount Δp such that ($\hbar$ is Planck constant):

$$\Delta q \Delta p \geq \frac{\hbar}{2} \tag{1}$$

Non-commutativity of quantum variables then puts limits on successive measurements, and impedes a continuous and independent determination of positions in space-time. For instance, successive independent measurements of position $q(t)$ and $q(t+\tau)$ of a free mass m are subject to a "standard quantum limit" [3]:

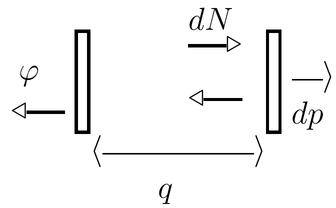
$$\begin{aligned} \Delta q(t+\tau)^2 + \Delta q(t)^2 &\geq 2\Delta q(t+\tau)\Delta q(t) \\ &\geq \frac{\hbar}{m}\tau = \Delta q_{SQL}^2 \end{aligned}$$

As extensively discussed in last years, successive independent measurements may not correspond to the best measurement strategy [4, 5]. As has been shown, when optimising the measurement strategy, the "standard quantum limit" can be beaten leading to actual limitations which are much lower [6, 7]. Here, we discuss those ultimate limitations which subsist as a consequence of the quantum noise induced by quantum field fluctuations in vacuum.

II. Probe field quantum fluctuations

We first briefly recall the quantum limits imposed by quantum fluctuations of the probe field in a measurement of the position of a mirror using interferometry. An interferometric measurement of position can be seen as a phase measurement for the probe field. On one output port of an interferometer, a signal is detected which varies like the difference of phase-shifts φ undergone by the probe field in different arms. For a monochromatic plane wave with wave-vector K_0, phase-shift variations directly provide an estimator for variations of one of the mirrors' position q:

$$\varphi(t) = 2K_0 q(t)$$



Back action during measurement with a probe field
Figure 1

Phase fluctuations of the probe field directly affect a position measurement. If the probe field intensity is increased, in order to improve signal-to-noise ratio by increasing the number of detected photons, then another source of noise, related to intensity fluctuations of the probe field ($I = dN/dt$, the time derivative of photon number N) also increases. As the probe field exerts a radiation pressure on the mirrors, intensity fluctuations affect the momentum p of the measured mirror and consequently its position (see Figure 1):

$$F = \frac{dp}{dt} = 2\hbar K_0 \frac{dN}{dt} = 2\hbar K_0 I$$

Coupled field and mirror provide an example of back action during measurement. The measured phase φ is also affected by fluctuations of its conjugate variable N. As fluctuations of conjugate variables are constrained by Heisenberg type inequalities, phase and intensity fluctuations of the probe field finally put limits on the allowed sensitivity in an interferometric measurement of position. When using coherent light as input fields, an optimum is reached when phase and intensity fluctuations provide equally important sources of noise, leading to the "standard quantum limit". Such limit relies on the independent character of the two conjugate sources of noise, which are related to the particular input fields used.

To discuss the general case, it is sufficient to analyse the effects of probe field fluctuations on measurement sensitivity in a linearised treatment of fluctuations [6]. During measurement, the probe field adds a noise q_n which is best described by its Fourier components:

$$q_n(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} q_n[\omega]$$

and which is the sum of a noise related to the phase of the probe field and of a noise related to the probe field intensity:

$$q_n = \frac{\delta\varphi}{2K_0} \ + \ \chi_{qq} \ 2\hbar K_0 \delta I \tag{2}$$

Noise frequencies are in mechanical range and are typically much smaller than probe field optical frequencies, so that phase and intensity variations appear as modulations of two conjugate quadrature components of the input

fields. Variations of position due to intensity fluctuations can be treated in linear response formalism, so that χ_{qq} describes the mirror's response to an applied force:

$$\delta q[\omega] = \chi_{qq}[\omega]\delta F[\omega]$$

For simplicity, one can consider that the mirror, of mass m, is mechanically bound with a proper frequency ω_0, and that all dissipative couplings can be summarised in a friction coefficient γ depending on the frequency:

$$\chi_{qq}[\omega] = \frac{1}{m(\omega_0^2 - \omega^2 - i\gamma\omega)}$$

Correlation functions of the noise added in the measurement are then characterised by their spectrum:

$$< q_n(t)q_n(0) > - < q_n(t) >< q_n(0) >= C_{q_nq_n}(t)$$

$$C_{q_nq_n}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} C_{q_nq_n}[\omega]$$

Constraints on spectra describing phase and intensity fluctuations, and correlations between phase and intensity, are put by Heisenberg type inequalities. They then determine lower bounds for spectra of the noise variable q_n. Several cases, corresponding to different measurement strategies must be distinguished.

For an antenna, like a gravitational wave detector, the limit fixed on sensitivity is best expressed by the minimal spectral energy density of the remaining noise. For probe fields with uncorrelated phase and intensity, Heisenberg inequalities for probe field variables take the usual form, similar to (1), for each frequency, and lead to the "standard quantum limit":

For a free mirror ($\omega_0 \simeq 0$):

$$\frac{1}{2} m\omega^2 C_{q_nq_n}[\omega] \geq \hbar m\omega^2 |\chi_{qq}[\omega]| \simeq \hbar \tag{3}$$

$$\Delta q_n^2 = C_{q_nq_n}(0) \simeq \Delta q_{SQL}^2 = \frac{\hbar}{m}\tau \qquad\qquad \tau \simeq \frac{\Delta\omega}{\omega^2}$$

($\Delta\omega$ is the detection bandwidth, providing a time parameter τ characteristic of measurement). The "standard quantum limit" is given by the modulus of the mirror's mechanical response function, i.e. essentially by its reactive part, and corresponds to a constant spectral energy density equal to $\hbar$.

In an optimal measurement, correlations of phase and intensity of the probe field must be adapted to the mechanical response function of the mirror. In other words, input fields must be used which minimise fluctuations for the particular combination of the two quadrature components which finally enters the output noise q_n (see (2)), at the expense of increasing fluctuations for the conjugate combination. Squeezing of input fields must also be realised

in different directions for different frequencies, as specified by the mechanical response function of the mirror. When considering such ideally prepared input fields, measurement sensitivity is then only limited by the dissipative part of the mirror's mechanical response:

$$\frac{1}{2}m\omega^2 C_{q_n q_n}[\omega] \geq \hbar m\omega^2 |\mathrm{Im}\chi_{qq}[\omega]| \simeq \hbar\frac{\gamma}{\omega} \tag{4}$$

Such limit is much lower than the "standard quantum limit" (3).

For optimized measurement strategy, quantum fluctuations still put a limit on sensitivity, which is determined by dissipation mechanisms [5, 6]. As a result of the coupling of the mirror to the probe field, that is to a system with infinitely many degrees of freedom, the mirror's dynamics necessarily contain a minimal dissipative component related to quantum field fluctuations. This dissipative part again contains an irreducible contribution due to vacuum field fluctuations, which can then be seen as fixing an ultimate limitation on measurement sensitivity.

III. Quantum fluctuations of position in vacuum

In this part, we discuss the dissipative contribution to the mirror's motion due to vacuum field fluctuations, i.e. in a state with no photons, and the resulting fluctuation-dissipation relations for the position of a mirror in vacuum. Quite generally, quantum field fluctuations induce quantum fluctuations of field stress-tensors, so that a mirror is submitted to a fluctuating radiation pressure (or a fluctuating force F) due to quantum fluctuations of the incident field:

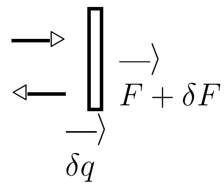
$$< F(t)F(0) > - < F >^2 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} C_{FF}[\omega]$$

As a result of general principles which govern motion in a fluctuating environment [8], the mirror is also submitted to an additional force when it moves. The motional force can be described within linear response formalism [9], by a motional susceptibility χ_{FF}:

$$\delta F[\omega] = \chi_{FF}[\omega]\delta q[\omega]$$

which is related to the generator of the perturbation (that is F for a displacement), and which satisfies a fluctuation-dissipation relation (see Figure 2):

$$2\hbar \mathrm{Im}\chi_{FF}[\omega] = C_{FF}[\omega] - C_{FF}[-\omega] \tag{5}$$



Fluctuating force at rest and force induced by motion

Figure 2

The mechanical response of the mirror to an applied force is then modified by its coupling to the probe field, and necessarily contains a dissipative part related to fluctuations of the incident field radiation pressure:

$$\chi_{qq}[\omega] = \frac{1}{m(\omega_0^2 - \omega^2) - \chi_{FF}[\omega]} \tag{6}$$

The dissipative contribution due to quantum field fluctuations depends on the input field state (5), but always includes a part which cannot be eliminated, and which is due to vacuum field fluctuations. Vacuum can also be seen as the state of thermodynamic equilibrium at zero temperature, so that it satisfies a further fluctuation-dissipation relation which completely determines fluctuation spectra from commutators only:

$$C_{FF}[\omega] = 2\hbar\theta(\omega)\mathrm{Im}\chi_{FF}[\omega]$$

$\theta(\omega)$ is Heaviside step function, ensuring that no excitations with negative energy can exist in vacuum. Vacuum quantum field fluctuations impose an irreducible dissipative contribution [10]:

$$\chi_{FF}[\omega] = im\gamma\omega = i\alpha\frac{\hbar\omega^3}{c^2} \tag{7}$$

where c is light velocity, and α is a dimensionless factor depending on the geometry and particular coupling between mirror and field (Lorentz invariance of vacuum also implies that the motional force does not contain any contribution proportional to the velocity).

Further fluctuation-dissipation relations characterise motion in vacuum. Under the effect of force fluctuations in vacuum, the mirror is submitted to a quantum Brownian motion which results in fluctuations of its position:

$$< q(t)q(0) > - < q >^2 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} C_{qq}[\omega]$$

The coupled system consisting of the mirror's position and field radiation pressure can be treated consistently within linear response formalism, and shown to also satisfy fluctuation-dissipation relations [11]. In particular, the position of a mirror coupled to vacuum fields satisfies both relations, that typical of linear response to an external perturbation:

$$2\hbar\mathrm{Im}\chi_{qq}[\omega] = C_{qq}[\omega] - C_{qq}[-\omega]$$

and also that typical of the zero-temperature limit of thermal equilibrium:

$$C_{qq}[\omega] = 2\hbar\theta(\omega)\mathrm{Im}\chi_{qq}[\omega]$$

In vacuum, positions have quantum fluctuations with spectra which are determined by the dissipative part of the mirror's mechanical admittance ((6) and (7)). Outside proper resonance frequencies, fluctuations are those induced by radiation pressure of vacuum fields. For a free mirror ($\omega_0 \simeq 0$):

$$\frac{1}{2}m\omega^2 C_{qq}[\omega] \simeq \hbar\theta(\omega)\frac{\gamma}{\omega} \tag{8}$$

$$\frac{\gamma}{\omega} = \alpha\frac{\hbar\omega}{mc^2} \ll 1$$

Coming back to the discussion of a position measurement, quantum fluctuations of the mirror's position include permanent fluctuations induced by radiation pressure of vacuum fields (due to field fluctuations with frequencies comprised between 0 and ω), and added fluctuations due to probe field fluctuations (field frequencies around the probe frequency cK_0). Taking these two noises into account again leads to a quantum limit for the sensitivity in an optimal measurement which is fixed by the dissipative part of the mirror's mechanical response (4) [11]. Hence, vacuum field fluctuations induce an ultimate quantum noise on position. Its order of magnitude is determined by the Compton wave-length corresponding to the reflector's mass. For macroscopic mirrors, (of mass greater than Planck mass $\sim 22\mu$g), the Compton wave-length becomes smaller than Planck length $\sim 1.6\ 10^{-35}$m, so that this limit is actually negligeable. At such level, quantum fluctuations related to gravitation must be taken into account in order to discuss actual quantum limits.

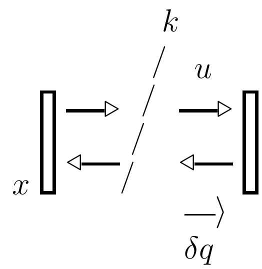
IV. Gravitational quantum fluctuations

Quantum fluctuations of gravitation must limit the determination of positions with a precision at the level of Planck length [12]. Indeed, quantum fluctuations of space-time metric affect length measurements. This perturbation can be described intrinsically, that is independently of a particular choice of a reference system, in terms of space-time curvatures. When propagating in space-time, a probe field registers curvature fluctuations. The main effect of metric perturbations with small wave-vectors k (when compared to probe field wave-vector K) is obtained in the eikonal approximation. Then, the probe field momentum, or wave-vector $K_\mu = K_0 u_\mu$, follows the law of geodesic deviation [13]:

$$\partial_\nu K_\lambda = \int_0^l R_{\lambda\mu\nu\rho}(x - u\sigma)K^\mu u^\rho d\sigma$$

Variations of momentum integrate curvature perturbations, described by their Riemann tensor $R_{\lambda\mu\nu\rho}$, encountered during propagation (along direction u, $u^0 = 1$) from the emitter to the receptor of coordinates x (l is the total propagation length and σ an affine parameter). In particular, phase shifts of the probe field can be obtained from frequency deviations (cK_0 being the time derivative of the phase). The corresponding expression coincides with the formula giving the red-shift induced by a stochastic background of gravitational waves [14]. It can also be written as a variation of the measured distance q (depending on time $t = x^0/c$) due to curvature fluctuations (of Fourier components $R_{\lambda\mu\nu\rho}[k]$) (see Figure 3):

$$\frac{1}{c^2}\frac{d^2\delta q}{dt^2} = \int_0^l d\sigma \int \frac{dk}{(2\pi)^4} e^{-ik.(x-u\sigma)} R_{0\mu 0\rho}[k] u^\mu u^\rho \tag{9}$$



Gravitational perturbation of probe field propagation
Figure 3

Quantum fluctuations of the metric can be treated like those of other elementary fields [15]. Linearised Einstein equations provide the graviton propagator, and hence, in agreement with fluctuation-dissipation relations, metric quantum fluctuations in vacuum [16, 17]. These are characterised by Planck length (G is Newton's gravitation constant):

$$l_p = \left(\frac{\hbar G}{c^3}\right)^{\frac{1}{2}} \sim 1.6\ 10^{-35}\mathrm{m}$$

At lowest order in l_p, metric vacuum fluctuations are determined by gravitational waves zero-point fluctuations. Corresponding space-time curvature fluctuations provide vacuum fluctuations which are invariant under gauge symmetries (i.e. metric transformations taking the form of those induced by changes of coordinates):

$$C_{R_{\lambda\mu\rho\nu}R_{\lambda'\mu'\rho'\nu'}}[k] = 16\pi^2 l_p^2 \theta(k_0)\delta(k^2)\times$$

$$\times(\mathcal{R}_{\lambda\mu\lambda'\mu'}\mathcal{R}_{\rho\nu\rho'\nu'} + \mathcal{R}_{\lambda\mu\rho'\nu'}\mathcal{R}_{\rho\nu\lambda'\mu'} - \mathcal{R}_{\lambda\mu\rho\nu}\mathcal{R}_{\lambda'\mu'\rho'\nu'})$$

$$\mathcal{R}_{\lambda\mu\rho\nu} = \frac{1}{2}(k_\lambda k_\rho \eta_{\mu\nu} + k_\mu k_\nu \eta_{\lambda\rho} - k_\mu k_\rho \eta_{\lambda\nu} - k_\lambda k_\nu \eta_{\mu\rho})$$

Riemann curvature fluctuations can also be determined from Lorentz and gauge invariance in vacuum, symmetry properties of Riemann tensor, correlations of Einstein tensor (which vanish) and normalisation to $\frac{1}{2}\hbar$ of spectral energy density.

When integrated along propagation of the probe field (see (9)), curvature fluctuations induce fluctuations of distances [18]:

$$C_{qq}[\omega] = \beta l_p^2 \frac{\theta(\omega)}{\omega}$$

where β is a geometric factor depending on the particular measurement technique used (one-way or round trip probing). One should note that for low frequencies, that is for frequencies well below Planck frequency, spectra of distance fluctuations induced by gravitational fluctuations only depend on the assumed effective behavior of gravitation at low frequencies (as described by Einstein theory) and conformity of vacuum fluctuations with fluctuation-dissipation relations. As a consequence of characteristic properties in vacuum, spectra of distance fluctuations only contain positive frequencies, so that when rewritten in space-time domain, distance correlations are not symmetric under exchange of their arguments. Geodesic distances then appear as non-commutative quantum variables.

For low frequencies, distance fluctuations induced by gravitational fluctuations take a form which is similar to that of fluctuations induced by radiation pressure. They however differ in their order of magnitude, gravitational fluctuations imposing a limit on space-time probing of the order of Planck length. For an optimal measurement, there results two regimes of ultimate space-time fluctuations, depending on the endpoint mass used.

For a "microscopic" mass, i.e. smaller than Planck mass:

$$m \ll m_p \qquad\qquad m_p = (\frac{\hbar c}{G})^{\frac{1}{2}} \simeq 22\mu\text{g}$$

radiation pressure fluctuations dominate, and the noise spectrum depends on the mass (see (8)):

$$C_{qq}[\omega] \simeq \lambda_c^2 \frac{\theta(\omega)}{\omega} \qquad\qquad \lambda_c = \frac{\hbar}{mc}$$

For a "macroscopic" mass, i.e. greater than Planck mass:

$$m \gg m_p$$

metric fluctuations dominate and the noise spectrum is universal:

$$C_{qq}[\omega] \simeq l_p^2 \frac{\theta(\omega)}{\omega}$$

Optimal space-time probing is thus obtained using "macroscopic" endpoint reflectors. Measurement of positions in space-time is then limited by

an ultimate quantum noise which is universal, as it only depends on universal constants through Planck mass. Quantum fluctuations of curvature in vacuum can thus be considered as giving rise to ultimate space-time fluctuations [18].

V. Conclusion

When optimising the input probe fields used, sensitivity in an interferometric measurement of position can be pushed beyond the "standard quantum limit", and becomes limited by dissipative mechanisms only. Vacuum field fluctuations induce quantum fluctuations of radiation pressure which fix an irreducible dissipative part for the mechanical response function of the measured mirror. As a result of fluctuation-dissipation relations, sensitivity in a position measurement can be seen to be limited by an ultimate quantum noise due to quantum field fluctuations in vacuum. For reflectors with a mass greater than Planck mass, the quantum noise induced by vacuum radiation pressure fluctuations becomes negligeable, and position fluctuations are dominated by metric vacuum fluctuations, which are universal and have an order of magnitude of Planck length. Optimal measurements in space-time are thus ultimately limited by quantum fluctuations of space-time itself, due to quantum fluctuations of space-time curvatures in vacuum. The non-commutative nature of space-time geometry already reveals itself at low frequencies, in the ultimate quantum limits imposed to space-time measurements.

References

[1] A. Brillet, T. Damour, P. Tourrenc, Ann. Phys. Fr. **10** (1985) 201.

[2] V.B. Braginsky, Usp. Fiz. Nauk **156** (1988) 93 [Sov. Phys. Usp. **31** (1988) 836].

[3] C.M. Caves, Phys. Rev. Lett. **45** (1980) 75; Phys. Rev. **D 23** (1981) 1693.

[4] H.P. Yuen, Phys. Rev. Lett. **51** (1983) 719.
C.M. Caves, Phys. Rev. Lett. **54** (1985) 2465.
M. Ozawa, Phys. Rev. Lett. **60** (1988) 385; Phys. Rev **A 41** (1990) 1735.

[5] V.B. Braginsky, F.Ya. Khalili, "Quantum Measurement" (Cambridge Univ. Press, Cambridge, 1992).

[6] M.T. Jaekel, S. Reynaud, Europhys.Lett. **13** (1990) 301.

[7] W.G. Unruh, in: "Quantum optics, experimental gravitation and measurement theory", eds. P. Meystre and M.O. Scully (Plenum, New York, 1983) p. 647.
A.F Pace, M.J. Collett, D.F. Walls, Phys. Rev. **A 47** (1993) 3173.

[8] A. Einstein, Ann. Physik **17** (1905) 549; Phys. Z. **10** (1909) 185, **18** (1917) 121.

[9] R. Kubo, Rep. Prog. Phys. **29** (1966) 255.
L.D. Landau and E.M. Lifschitz, Cours de Physique Théorique, Physique Statistique, première partie (Mir, Moscou, 1984) ch. 12.

[10] M.T. Jaekel, S. Reynaud, Quantum Optics **4** (1992) 39.

[11] M.T. Jaekel, S. Reynaud, J. Phys. I France **3** (1993) 1.

[12] J.A. Wheeler, Ann. Phys. **2** (1957) 604.

[13] V.B. Braginsky, M.B. Mensky, Pisma Zh. Eksp. Teor. Fiz. **13** (1971) 585 [Sov. Phys. JETP Lett. **13** (1971) 417].

[14] B. Mashhoon, L.P. Grishchuk, Astrophys. J. **236** (1980) 990.

[15] R.P. Feynman, Acta Physica Polonica **24** (1963) 711.
S. Weinberg, Phys. Rev. **138** (1965) B988

[16] R. Utiyama, B.S. de Witt, J. Math. Phys. **3** (1962) 608.

[17] M.T. Jaekel, S. Reynaud, Ann. Physik **4** (1995) 68.

[18] M.T. Jaekel, S. Reynaud, Phys.Lett. **A 185** (1994) 143.